13. Problem sheet for Set Theory, Winter 2012

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Problem 47.

(a) Determine the class of infinite cardinals κ such that the intersection of finitely many cub subsets of κ is cub.

(b) Determine the class of infinite cardinals κ such that Fodor's Lemma holds for κ.

Problem 48. Suppose $(P, \leq)$ is a partial order of size κ without maximal elements. A set $C \subseteq P$ is *unbounded in P* if $\forall x \in P\ \exists y \in C\ y \not\leq x$. Show that there is a unbounded wellordered chain $C \subseteq P$.

Problem 49.

(a) If $2^{\aleph_n} < \aleph_\omega$ for all $n < \omega$, show that $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0}$.

(b) Let $f(\gamma) = \delta$ with $\prod_{\alpha<\gamma} \aleph_\alpha = \aleph_\delta$. Under GCH show

  (1) $f(\gamma) = \gamma + 1$ for $\gamma \in Lim$ using König's theorem.

  (2) $f(\gamma + 1) = \gamma + 1$ for $\gamma \in Lim$.

  (3) $f(\alpha + 2) = \alpha + 1$ for all $\alpha \in Ord$.

Problem 50. Suppose κ is a regular infinite cardinal. Recall the definition of filters in Definition 75.

- A filter F on κ is *free* if $\{\alpha\} \notin F$ for all $\alpha < \kappa$.
- A filter F on κ is an *ultrafilter* if $X \in F$ or $\kappa \setminus X \in F$ for all $X \in \mathcal{P}(\kappa)$.

Show

(a) A set $F \subseteq \mathcal{P}(\kappa)$ is a κ-complete ultrafilter on κ if and only if the map $\mu_F : \mathcal{P}(A) \to \{0, 1\}$ with $\mu_F^{-1}(\{1\}) = F$ is a κ-additive measure.

(b) If F is a κ-complete free ultrafilter on κ, then $card(X) = \kappa$ for all $X \in F$.

There are 6 points for each problem. Please hand in your solutions on Monday, January 21 before the lecture.